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Artificial Modification of the Earth's Radiation Belt*,†

S. F. Singer‡

Summary

The fundamental theory of the behavior of charged particles, such as electrons and protons, in a magnetic field was set out by the Norwegian astrophysicist Störmer, and then developed more fully by the Swedish astrophysicist Alfvén. His theoretical ideas were applied by us to a hypothesis that charged particles from the sun and from cosmic rays could be trapped in the earth's magnetic field. This hypothesis was developed originally to explain the origin of magnetic storms and aurora. It received striking support by Van Allen's discovery of the "radiation belt" which must consist of such trapped particles. From theory we can calculate the length of time for which the particles stay trapped and even make suggestions as to their origin, but to get greater precision in theoretical predictions it is essential to do experiments. One of the most interesting experiments which suggests itself is that of injecting artificially accelerated particles directly into the radiation belt region. One promising method would consist of carrying a small electron accelerator aloft in either a satellite or even a high altitude rocket. The most direct place for conducting the experiment is near the equator at an altitude of from 600 to 1000 miles. The particles which are injected, preferably electrons of the order of 2 Mev, would rapidly diffuse in a type of "napkin ring" region around the earth. This diffusion only requires a few seconds since the particles move with nearly the speed of light. Electrons then stay trapped in this napkin ring region, diffusing out of it very slowly. The primary reason for their outward diffusion may be collisions with the very rarified atmosphere which is still present even at these high altitudes. Therefore by studying the number of remaining electrons from day to day we can learn more about the density of the atmosphere in a region of space where densities can normally not be measured. The simplest way to determine this density is by firing sounding rockets daily through this napkin ring region and determining the electron intensity profile from day to day and see how it decays. We would expect such a region to last for a few days to a few weeks, depending on the altitude and depending on what actual air densities exist at these altitudes.

The injection apparatus would only weigh of the order of 500 pounds including its power supply, and could provide us with a very substantial increase in the radiation belt intensity in a small region. The reason for this striking increase with such small equipment is again due to the fact that the earth's magnetic field has such good trapping properties. The sounding rockets which are sent up after the injection to check up on the fate of the electrons can be quite simple and need only to carry Geiger counters with associated circuits weighing on the order of a pound or less. A project of this type is therefore feasible immediately.

Historical Introduction

The earth's magnetic field may to a good approximation be represented by that of a dipole. More than fifty years ago Carl Störmer, a Norwegian astrophysicist, became interested in the problem of the motion of a charged particle in such a dipole field (1). He developed a very elaborate theory which may be summarized as follows: A charged particle starting off far from the earth may or may not reach a point under consideration. In general, there will be some directions of incidence which are allowed. These directions are therefore contained within the "allowed cone" while the rest of the solid angle may be called the "forbidden cone" since particles of a certain energy *coming from infinity* cannot reach the point under consideration in that direction. (Of course, for particles of very low energy all directions are "forbidden," while for particles of extremely high energy, which travel through the field without much deflection, all directions are "allowed.")

Störmer undertook this study in order to explain the origin of the aurora, assuming that they were produced by particles originating at the sun and then entering the earth's magnetic field. However, it turned out that his studies were more appropriate to the behavior of the much higher energy cosmic rays. Störmer and the workers following him mostly paid attention to those particles which could reach the earth from infinity. But what if we trace back the trajectory of a particle along a forbidden direction? It turns out that these particles could not have come from infinity, but that their orbits are "trapped," i.e., confined to the vicinity of the earth. Störmer, in fact, showed that there are regions around the earth in which particles could move but that these regions have no physical

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connection to infinity, and he concluded therefore that they are of no interest (Figure 1).

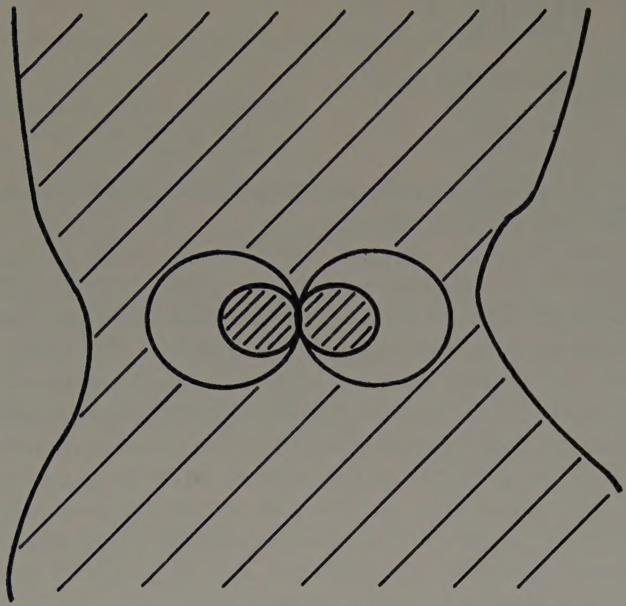

FIG. 1. Diagram showing the allowed and forbidden spaces in the vicinity of the Earth's dipole according to Störmer. Small changes of the dipole field can open up to infinity the allowed region in the immediate vicinity of the dipole; this region extends narrow funnels into the Earth's atmosphere in the auroral zones; for particles of very low energies, encountered in solar corpuscular streams, the simpler peturbation methods of Alfven can be used to discuss the problem of trapping in the Earth's dipole field.

First to consider the possibility that there might be particles in trapped orbits seems to have been Hannes Alfvén, a Swedish astrophysicist, well known for his development of the science of magnetohydrodynamics. Alfvén investigated the orbits of cosmic rays in the vicinity of the sun, assuming the sun to have a dipole field. He showed by simple considerations that this dipole field could not be perfect because the earth possessed a magnetic field which would perturb the sun's field near the earth. Therefore the earth could scatter particles into these forbidden regions and the particles might remain there for long periods of time (2). This idea was developed in greater detail and with great elegance by Professor John Wheeler of Princeton University and his students (3). However, the idea was not taken up in a practical way because further work showed that the sun probably did not have a dipole field and that in any case the motion of the conducting interplanetary gas would disturb the sun's magnetic field considerably.

In the meantime, laboratory experiments had been performed which clearly showed the existence of particles in the forbidden regions. Birkeland, in Norway, (4) at the end of the last century was first to develop the potentialities of the "terrella," a model earth having a magnetic dipole field. This was followed many years later by very precise work in Alfvén's Institute by Malmfors (5), Brunberg and Dattner (6), and by Block (7), leading to model experiments on the trajectories of cosmic rays and on the origin of the aurora. Willard Bennett (8) at the Naval Research Laboratory carried out an especially elegant demonstration of the trajectories of electron streams in the vicinity of a terrella. His photographs of the "Störmertron" show very clearly the existence of particles in forbidden regions around a dipole.

The existence of trapped particles in the earth's magnetic field (9) seems to have been considered first in 1956. Even though a single particle could not enter into the forbidden region (as shown by Störmer's theory), Singer reasoned that if a large number of particles were to arrive from the sun, their collective action could perturb the strict dipole field sufficiently to allow entry into the trapping regions.

By perturbing the magnetic field, particles are scattered into the normally forbidden regions so that after a solar corpuscular cloud has passed the earth, a certain number of particles may remain in this region. The motion of the particles was discussed and it was shown that they would not immediately hit the earth but, instead, remain trapped for long periods of time (9) and would be removed only by interactions with the very tenuous atmosphere or by the scattering effects of magnetic field inhomogeneities. It was also suggested that cosmic ray albedo could make a contribution to this trapped radiation (9).

It must be understood that the assumption of particles in the forbidden regions was purely a hypothesis and was developed mainly to explain magnetic storms and aurora. Subsequently, Singer developed a more detailed theory for the main phase of magnetic storms which makes use of the drift of these trapped particles to explain the formation of a ring current (10). But since the occurrence of a ring current was not certain and since in any case its mechanism was not understood, the hypothesis of the existence of trapped particles in the earth's dipole field was still unsupported.

An important advance from an experimental point of view was made by Van Allen and his colleagues who measured for the first time the incidence of charged particles in the auroral zone (11). Those particles were presumed to be electrons. Although the existence of particles had been suspected for a long time as a source of the aurora, this constituted the first direct measurement. Now if the hypothesis of trapped particles was correct, then one should be able to see these particles also at lower latitudes but at higher altitudes, namely along the same line of force (see Figure 2). Accordingly, a proposal was made to detect auroral particles by means of a high altitude rocket launched at low latitudes (12). This experiment was one of the main objectives of the U. S. Air Force FAR SIDE project (13), an extreme high-altitude probe which was launched in October 1957 from Eniwetok and penetrated well into the radiation belt region.

Van Allen's dramatic discovery of a layer or belt

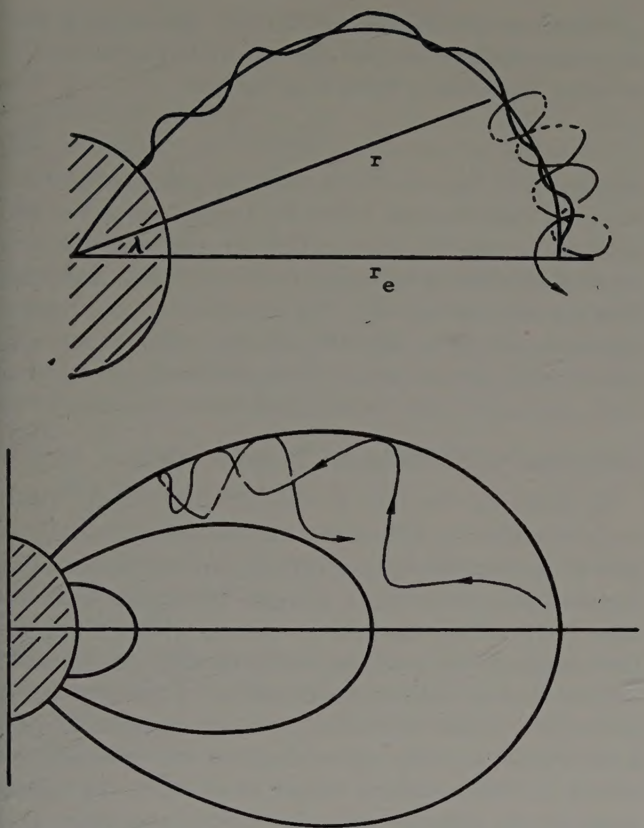

FIG. 2. Schematic indication of the reflection and trapping of a charged particle by the earth's magnetic field. The particle's velocity makes a pitch angle α_0 to the magnetic line of force in the equatorial plane. As the particle moves into a region of stronger magnetic field (lines closer together), its pitch angle increases until it reaches 90° at which point the particle is reflected and moves into the opposite hemisphere where it will again be reflected from a similar turning point. With a smaller value of α_0 the particle will approach closer to the earth's surface. For extremely small values of α the particle can enter the earth's atmosphere. With $\alpha_0 = 90°$ the particle remains in the equatorial plane. Both soft and hard radiation belt particles are trapped in this manner.

of radiation surrounding the earth (14) (which became evident through a Geiger counter in the first Explorer satellite) gives us now direct evidence for the existence of trapped particles in the vicinity of the earth.

In what follows we will be discussing a summary of the trapped particle theory and some of its applications, in particular the possibility of modifying the radiation belt. We want to discuss in detail the possibility of injecting particles into the earth's magnetic field in order to study the storage properties of the magnetic field and in order to measure the atmospheric density at very high altitudes.

Theoretical Treatment: The Trapping of Particles

The theoretical study of the radiation belt revolves on the question of how particles may enter or be released into the trapping regions and in what manner and for how long the particles move inside this region.

Turning to the second problem, the motion of particles in the magnetic field, we find here interesting developments reaching back to nearly 100 years. A magnetic dipole field, such as that of the earth, can be represented as follows. The equations below give the magnetic field strength B as a function of distance r from the dipole center and as a function of the polar angle λ measured from the equatorial plane.

$$B = \frac{M}{r^3}(1 + 3\sin^2\lambda)^{1/2} \tag{1}$$

where $M = 8.1 \times 10^{25}$ gauss-cm^3, the earth's dipole moment.

The lines of force which indicate everywhere the direction of the field are then given by an equation of form

$$r = r_e \cos^2\lambda \tag{2}$$

where r_e is the distance from the dipole center where the line of force intersects the equatorial plane. Proceeding along a line of force starting from the equatorial plane it can be seen by inspection that the lines of force crowd together as one approaches the earth and that therefore the field becomes increasingly stronger. It is not possible to express the magnetic field strength B as a function of distance along the line of force in closed form; however, various approximate formulas can be developed to describe B (15). In polar coordinates, along a given line of force

$$B = \frac{M}{r^3}\frac{(1 + 3\sin^2\lambda)^{1/2}}{\cos^6}. \tag{3}$$

A charged particle moving in a magnetic field **B** with velocity **V** experiences the well-known Lorentz force $(e/c)(\mathbf{V} \times \mathbf{B})$. If we decompose its velocity vector **V** into a velocity $V_\parallel$ parallel to **B** and $V_\parallel$ perpendicular to **B** we will find the particle moving along a helical trajectory whose pitch angle α is given by $\tan\alpha = V_\perp/V_\parallel$. The particle thus advances along the field with velocity $V_\parallel$, meanwhile gyrating about the line of force. The period and radius of gyration are given by

$$t_c = \frac{2\pi mc}{eB}; \tag{4}$$

$$\rho = \frac{mV_\perp c}{eB} \tag{5}$$

It was first realized by the French mathematician Poincaré (16) that a charged particle which moves in an inhomogeneous magnetic field and approaches a region of increasing strength of the magnetic field can be reflected back into the region of weaker field, i.e., that $V_\parallel$ decreases to zero and changes its sign. Störmer (1) carried out many numerical calculations of particle orbits which clearly exhibit this feature of reflection by the stronger magnetic field. However, it was Alfvén who first showed that this property of the motion of a particle could be understood in terms of an approximate conservation law (17), namely the conservation of the particle's magnetic moment μ, defined as the

current generated by its gyration times the area enclosed. Thus

$$\mu = \frac{e}{c}\frac{1}{t_c} \cdot \pi\rho^2 = \frac{\frac{1}{2}mV_\perp^2}{B} = \frac{1}{2}mV^2\frac{\sin^2\alpha}{B}. \qquad (6)$$

Alfvén showed that this quantity remains constant to a good approximation in a magnetic field which varies smoothly (17). If the magnetic field possessed intensity variations which are severe over a distance shorter than the particle's radius of curvature, then the magnetic moment of the particle would no longer stay constant; i.e., when

$$\rho \text{ grad } B/B > 1. \qquad (7)$$

This latter case of course holds for the motion of cosmic rays in the earth's dipole field since their radius of curvature can be very large. As a matter of fact, no proper theory has as yet been developed to indicate precisely how the magnetic moment may vary in a dipole field and how rapidly this approximate constancy is lost. Numerical calculations indicate that the magnetic moment is conserved even upon many reflections to a surprising degree (18), but for long life times such questions have to be considered with much more precision. However, in what follows we have assumed strict conservation of the magnetic moment.

Alfvén also pointed out (17) that time variations of the magnetic field with a time scale shorter than the gyration time of the particle could also change its magnetic moment; i.e., for

$$t_c \frac{dB}{dt} \cdot B > 1. \qquad (8)$$

We shall not consider these effects in the development of the theory but return to them only later.

Keeping in mind that the kinetic energy of a particle must remain constant in its motion in a time-invariant magnetic field we see that $\sin^2 \alpha/B$ must remain constant. It can therefore be seen that $\sin \alpha$ reaches its maximum value of 1 (and α becomes 90° and $V_\parallel$ becomes zero) at a certain value of the magnetic field B. The particle cannot move beyond this point since $\sin \alpha$ cannot increase beyond 1; therefore the particle will be reflected at this turning point. The location of the turning point depends in the first instance on the particular line of force along which the particle is moving and secondly on its initial pitch angle α_0 (defined as the pitch angle of the particle when it crosses the equatorial plane of dipole).

We have therefore a picture of particles spiraling along lines of force from the northern into the southern hemisphere to be reflected there, returning to the northern hemisphere, being reflected at a symmetrical turning point, and therefore remaining trapped in the earth's magnetic field. Particles with very small pitch angle can of course reach much lower altitudes and may therefore escape into the dense atmosphere almost immediately (see Fig. 2).

Superimposed on this oscillating motion is a slow azimuthal drift of the particles which may give rise to a current. The drift velocity is given by

$$V_d = kV\rho/r \qquad (9)$$

where k is of the order of 1, and depends on the value of α_0. In fact we were led to the hypothesis of trapped particles in order to explain through this drift the ring current responsible for magnetic storms (10). As of this date the existence of such a current has not been verified experimentally although the satellite observations certainly verify the existence of trapped particles, and the drift associated with them follows from the theory.

Lifetimes of Particles in Trapped Orbits

In order to find the lifetime of particles in these trapped orbits (and from this their intensity), we must discuss how particles once trapped can be released. The physical ideas were again discussed in earlier publications (10) and are basically connected with a change in pitch angle. If the pitch angle of a particle can be made smaller then it will travel further along a line of force, and reach deeper into the dense atmosphere; having gone deeply into the atmosphere, it will lose enough energy so that it will no longer be effective. The pitch angle can be changed basically by two methods (10). First, by collisions with atoms or ions of the outermost atmosphere of the earth, and secondly, by scattering from discontinuities and short-scale time-variations of the earth's magnetic field (as indicated by equations (7) and (8)). Which of these two processes is most important depends on the particular situation which we discuss. For example, for low altitudes at the equator, the first process is likely to be most important since at low altitudes the atmospheric densities are still quite high. Also near the equator the earth's magnetic field is "quiet" as compared to the state of the field in the auroral region. In the auroral region at high latitudes, and along the same line of force at high altitudes, the magnetic field is more lilely to be disturbed by the continuing incidence of solar corpuscular streams and by the turbulence contained in these streams. In fact, bunching of the trapped particles may themselves originate magnetic field fluctuations which propagate as hydromagnetic waves and serve to accelerate some of the particles while releasing them through the change of their pitch angle. While this phenomenon may be of importance to the explanation of the aurora (19), it will not concern us in this paper.

Before discussing any theory for the origin of the radiation belt, it is important to calculate the lifetime of particles in trapped orbits. This can be done with some precision for particles at low altitudes near the equator. There, as indicated earlier, the residual atmosphere is likely to provide the most important mechanism for limiting the particle's lifetime. We have investigated this point very carefully and found that relativistic electrons (i.e., electrons with energy greater

than the electron rest mass of 0.5 Mev) are scattered in interactions with atoms, ions and other electrons, and that this angular scattering produces a type of "wandering" in pitch angle. As shown in Ref. (20), this "random walk" in pitch angle can be described by a diffusion equation which leads then to a means for estimating the rate of escape of the particles from the trap. On the other hand, for protons having an energy of about 100 Mev the main process is found to be energy loss by collisions (21). It has been shown that this will decrease their energy to a point where the protons are no longer detectable and therefore are no longer of experimental interest.

Origin, Intensity, and Altitude Distribution of the Radiation Belt

When the high intensity radiation was discovered in Explorer I satellite, it was at first thought that this radiation might be due to the trapped particles responsible for the magnetic storms and aurorae. At high latitudes or large altitudes solar corpuscular radiation should certainly be the predominant source. However, it is difficult to see how this radiation can diffuse to low altitudes at the equator; at least no quantitative theory has as yet been given for such diffusion. Therefore, we have investigated other injection mechanisms. We have worked out in some detail a possible mechanism based on cosmic ray albedo neutrons, which could provide trapped particles in the equatorial and low latitude regions (21). The same idea has been suggested independently in the Soviet Union by Lebedinsky and Vernov (22); and by Rothwell, Gold (23) and Kellogg (24). Kellogg's calculations, however, give values for lifetimes and intensities which differ considerably from our values.

The albedo neutron injection works as follows: Primary cosmic rays smash into the earth's atmosphere in the vicinity of 20 to 25 km (70,000 feet) and disintegrate atmospheric nuclei. Neutrons are released with energies up to several 100 Mev; many of them travel upwards and are therefore called "cosmic ray albedo."

Being uncharged, the neutrons travel along straight lines right through the earth's magnetic field out into space, but free neutrons are radioactive. A small fraction of the fast neutrons, about 10^{-13} per cm or about one in a million in 100 miles, change into protons and electrons while traveling through the field. Compared to the protons which have an energy of the order of the neutron energy, the lifetime of the electrons is rather small. The protons immediately spiral around the line of force, a fraction having a small pitch angle are lost immediately as they get too deep into the atmosphere, but the rest are trapped.

The essential part of the calculation can be understood as follows: The concentration N of trapped particles follows from a balance between the injection rate Q and the rate of removal; the latter can be expressed in terms of a lifetime T and is found to be determined mainly by collisions with atmospheric atoms and ions. Roughly then, we find

$$N \text{ per cm}^3 \sim Q \text{ (per cm}^3 - \text{sec)} \times T \text{ (sec)} \quad (10)$$

But the experimentally determined quantity is the flux J, given by J (per cm^2 − sec) $\sim N$ (per cm^3) $\times$ V (cm/sec). A typical injection rate is 10^{-13}/cm^3-sec (for protons near 50 Mev) i.e., 1 particle is injected into a given cm^3 of space every 300,000 years. But the theoretically calculated "lifetime" is so long that the particle concentration builds up to about 10^{-7} per cm^3 at 1000 km (600 miles).* With a velocity of $\sim 10^{10}$ cm/sec the flux is $\sim$1000.

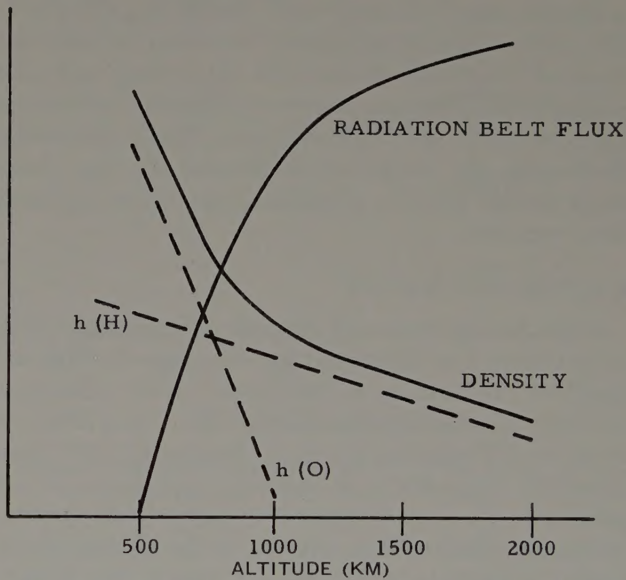

FIG. 3. Radiation belt intensity and atmospheric density and composition at the equator. The atmospheric density was derived from satellite data, the low altitude values from the atmospheric drag, and a high altitude point at 1900 km from the radiation belt intensity and the theory of Ref. 21. The exosphere critical level is found at 530 km; above this level collisions between atoms can be neglected and the temperature remains constant at 1500° K. Oxygen, because of its smaller scale height falls off very rapidly with altitude until at about 1000 km hydrogen predominates. This region then we call the "outer exosphere". Using these new values for the atmospheric density the radiation belt intensity can be calculated as a function of altitude and is shown in the graph.

The lifetime which is calculated is found to be inversely related to the atmospheric density. Since the latter decreases with altitude, the lifetime and therefore intensity of the radiation increase with altitude. The radiation flux (in particles per cm^2-second-steradian) is found to vary with atmospheric density d as $J \sim 10^{-14}/d$ at low altitudes near the equator (21). With a radiation belt flux of 3000 at 1900 km given by Explorer IV (22) we can therefore derive the density of the atmosphere at 1900 km as 2.5×10^{-18} gram per cm^3, i.e., an H-atom concentration of about 1.5×10^6

* The situation is analogous to a water tank which has a very tiny leak; then a small injection rate can build up a high level.

per cm^3. Combining with density values at lower altitudes from satellite orbit data we can construct a model of the exosphere. Its temperature is 1500° K. The critical level is found to be at 530 kilometers and transition to the "outer exosphere" (where neutral hydrogen predominates) occurs at 1000 kilometers. (See Figure 3).

Using these values for the atmospheric density calculated from the one observed radiation belt point, we can now calculate also the variation of radiation intensity with altitude. For the *cosmic ray albedo* radiation belt a maximum is reached at the equator at about $1\frac{1}{2}$ earth radii (near 6000 miles) beyond which its intensity drops. The radiation belt which is of solar origin and connected with aurora should reach its maximum near 5–6 earth radii (20,000–24,000 miles). The belt's intensity ultimately decreases because the trapping properties of the field deteriorate, and disappears when the interplanetary magnetic field takes over, probably near 10 earth radii. These theoretical predictions can be checked experimentally; the moon shots should give an excellent opportunity for such measurements.

Experimental Results

Experimental results on the radiation belt have been obtained by Van Allen and his colleagues in Explorer satellites (14) and by S. N. Vernov, A. E. Chudakov and their colleagues in Sputnik III (26). (In a different Sputnik III experiment, V. I. Krasovsky (27) has measured large fluxes of electrons and photons, especially in the auroral zone, and has pointed out the great importance of this radiation to the heating of the upper atmosphere in the auroral zone.) The experimental results have as yet not been presented in a definitive way.

As of this date the nature of the radiation belt particles has not actually been determined. However, in our view, there are in reality two radiation belts coexisting at the same time: (1) the *hard* radiation belt, just discussed, which is of cosmic ray origin, and can be calculated with great accuracy from cosmic ray data and theory. (2) In addition we also have the *soft* radiation belt which is of solar origin and connected with magnetic storms and aurorae. However, its location is different; it surrounds the hard radiation belt like a halo, although there must be a region of overlap (see Figure 4). As a consequence the radiation belt picture changes at different latitudes and different altitudes. Table 1 indicates schematically the properties of the hard and soft radiation belt of the earth.

The energy spectrum to be anticipated is shown in Figure 5. The high energy portion is the *hard* radiation belt, the low energy portion is due to the soft radiation belt. The intensity of the soft belt will vary depending on location and time; for example, it will be absent at low altitudes near the equator and will be a maximum in the auroral zone following intense solar activity. As a consequence of this complicated mixture of spectra, experiments with thin absorbers are likely to give inconsistent results. Since the hard radiation belt should be quite constant with time, the use of shields having a thickness of about 10 g-cm^{-2} or more, should give reproducible flux values.

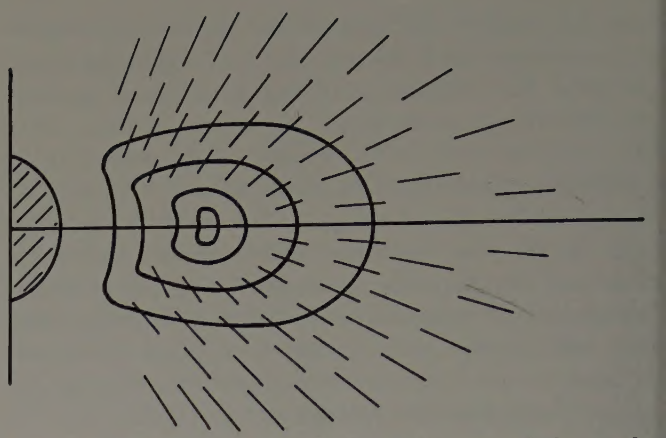

FIG. 4. Distribution of radiation belts around the earth, derived from the theory in Ref. 21 and 30. The lines emanating from the earth indicate the lines of force of the earth's magnetic field. The closed lines give contours of equal radiation intensity for the hard radiation belt, which is of cosmic ray origin. Its maximum is in the vicinity of 1.5 earth radii. Its intensity begins to increase at about 600 km (according to the theory) and may reach out as far as 10 earth radii (40,000 miles) at the equator. Note that the regions around the poles are clear. The soft (auroral) radiation belt is indicated by cross hatching and overlaps somewhat with the hard radiation belt at higher latitudes and larger altitudes. The maximum of the auroral particle radiation belt should occur between 4 to 6 earth radii and depending on the detector and shielding used may exceed in intensity the hard radiation belt.

Artificial Modification of the Radiation Belt

One may take advantage of the excellent trapping properties of the earth's field in two different ways. If the lifetime is as long as one calculates based on atmospheric scattering only, then it may be possible to substantially decrease the intensity level in the belt by providing additional absorbing bodies such as large satellites (28).

However, before carrying out any large-scale experiments, one would like to study the trapping properties of the field experimentally. For this purpose, it would be most interesting to inject particles into the magnetic field directly at high altitudes and trap them (28). Electrons are better suited for the injection than protons since they are much easier to generate and since their radius of curvature is very small. For this reason, they become quite insensitive to magnetic field fluctuations and to questions of the constancy of the magnetic moment. (Compare equations (7) and (8)). For example, a 2 Mev electron has a radius of curvature at an altitude of 1000 km above the equator, which is only $\frac{1}{3}$ km as compared to 70 km for a 100 Mev proton. The injection is best carried out by means of a linear accelerator which is carried aloft in a satellite; even a high altitude rocket may be adequate.

It turns out that an injection time of only a few

TABLE 1

Properties of Earth's Radiation Belts

	Hard (cosmic-ray)[a]	Soft (Auroral)[b]
Origin	From the decay of cosmic ray albedo neutrons which come out of the earth's atmosphere.	From Solar Corpuscular streams with subsequent acceleration near the earth.
Nature	Protons between 10–400 Mev	Electrons and protons of less than 1 Mev
Shielding	Difficult to absorb and shield.	Easily absorbed
Location	Equator and low latitudes Close to the earth	Auroral latitudes Far from the earth
Time dependence	Constant in time	Variable, increases when sun is active

[a] From Reference (21).
[b] From Reference (10).

minutes is quite sufficient to give an increase of the electron intensity observable over the natural radiation belt intensity (28).

The way in which such an experiment might work is as follows: The accelerator should be as small and as light as possible. This suggests a linear accelerator of the type that uses a klystron as an electron source and also as a means of supplying energy to a wave guide which accelerates the electrons. The instrument can be constructed to be extremely light-weight since the vacuum system can of course be dispensed with. This leaves only the electron gun, the tube structure, focusing magnet, and wave guide. By supplying a small magnet the electron beam can be doubled up and the whole accelerator can be folded into a smaller space. The power supply requirements can be estimated reasonably as about 50 kW for a few minutes. Applying a normal weight factor for batteries, we derive a total payload weight of about 500 pounds which is surprisingly small compared to what one might expect from laboratory types of electron accelerators.

The mean output current of such an accelerator would be about 500 microamperes giving therefore 3×10^{15} electrons per second. If we inject over an altitude interval of 10 km, then the volume into which the electrons are released and diffuse into is of the order of 10^{24} cm^3. Its shape resembles a "napkin ring."* The mean injection rate thus becomes 3×10^{-9} electrons per cm^3 per second. For simplicity we may assume that the injection is isotropic, i.e., the satellite or rocket is tumbling so that electrons are injected into all directions (it should be noticed here that it is not necessary to provide a well-defined electron beam, and this again saves considerable weight in the construction of the accelerator). Hence, about one-half of the electrons remain trapped.* Since the concentration of particles in the natural radiation belt is of the order of 10^{-7} per cm^3, we can double the radiation intensity after about 100 seconds of operation.

After building up the intensity in this manner, it will decay away slowly to its original value after the accelerator is turned off. The trapping volume will change from a napkin ring into a ring of more nearly circular cross-section (Fig. 5). The main point of the experiment now is to study the rate and manner in which the radiation intensity falls off after the injection.*

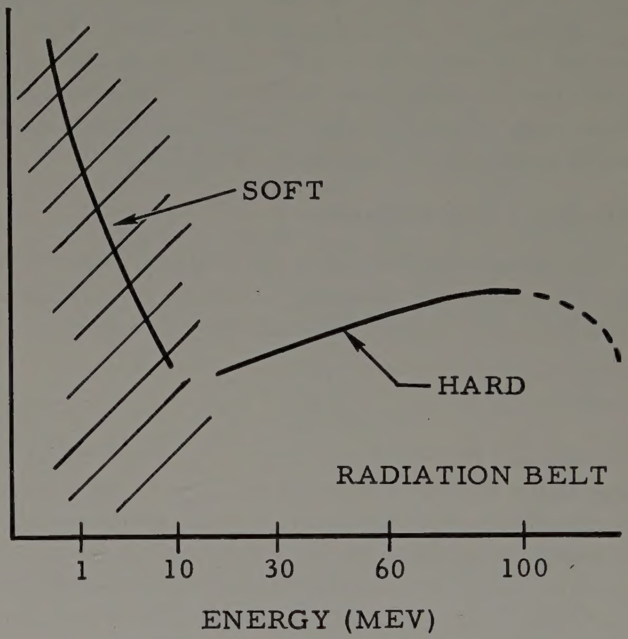

Fig. 5. Differential energy spectrum of the radiation belt flux. The differential flux is shown as a function of the proton velocity β. Note that the differential flux actually increases with β until the supply spectrum cuts off, or until the trapping properties of the magnetic field deteriorate (indicated by dash-line) The corresponding kinetic energies and ranges are shown giving the thickness of lead required to stop a proton of given energy. At very low energies we find the soft radiation belt, its contribution depending on latitude, altitude, and probably time. Its uncertain variation is indicated by cross-hatching. In any case its energy spectrum will fall off very steeply.

After the electrons are released, we would then check on the artificial radiation belt, perhaps once a day, by means of simple sounding rockets which traverse the region of the belt and determine the profile of electron intensity with altitude. From a comparison with the theoretically predicted variation one may then expect to study the trapping properties of the earth's magnetic field, therefore the mean lifetime of the electrons and thereby the density of the atmosphere at various altitude levels.†

* The details of the calculation are given in the mathematical appendix.

† We have confined our discussion entirely to an experiment in which the electrons are released and observed at or near the equatorial plane. Quite independently, N. C. Christofilos has considered the injection of electrons for very much the same purposes but has elaborated on a somewhat different technical approach (29). Basically, however, our ideas are very similar.

Conclusions

Modification of the radiation belt by artificial injection of particles promises to become a valuable tool for investigations of upper atmosphere densities and of the trapping properties of the earth's magnetic field. This in turn will lead to a better understanding of the natural radiation belt and of its connection with magnetic storms and aurora. It should be noted that such an experiment of artificial injection of electrons can be carried out immediately with available rockets and fairly direct modifications of existing instrumentations. Once such experiments are carried out, many other useful applications may suggest themselves.

Mathematical Appendix

We consider injection of 2 Mev electrons near 1000 km altitude at the equator. We derive the magnetic field $B = M/r^3$ and the radius of curvature ρ as

$$\frac{pc}{eB}$$

Numerically $B = 0.2$ gauss and $\rho = 3.3 \times 10^4$ cm. It is seen that $\rho \ll B/\text{grad}\, B = r/3 \sim 2.5 \times 10^8$ cm so that the condition of the constancy of the magnetic moment is probably well satisfied.

If the line of force intersects a given altitude level at a latitude λ, then the range of pitch angles of particles which stay trapped above this level is $\pi/2 \geqq \alpha_0 \geqq \alpha_c$, where $\alpha_c = \arcsin [\cos^3 \lambda(1 + 3 \sin^2 \lambda)^{-1/4}]$. If we set this lowest level one scale height H_0 below r_e then with $r_e = 7.37 \times 10^8$ cm and $H_0 \sim 10^7$ cm, and using the polar equation for a line of force, we find $\cos^2 \lambda = 1 - H_0/r_e$ and $\lambda = 6.7°$. This gives for $\alpha_c = 64°$. (The fraction of the solid angle filled with trapped particle intensity is approximately $\cos \alpha_c = 0.44$.) Thus the latitude (North–South) dimension will be approximately $2r\lambda = 2 \times 7.37 \times 10^8 \times .116 = 1.71 \times 10^8$ cm. The radial dimension will be at least $\sim 4\rho = 1.32 \times 10^5$ cm, but can be controlled by controlling the time of release of the electrons. The azimuthal dimension will be $2\pi r = 4.63 \times 10^9$ cm. Thus the volume will be $\sim 10^{24}$ cm^3 to begin with if we inject over a 10 km altitude interval near the equator.

To calculate the trapping time we set up a diffusion equation for the trapped particles:

$$\frac{\partial n}{\partial t} = \text{div}\, j + q$$

One-dimensional Treatment

The concentration n, diffusion current j and source function q are all functions of pitch angle α, energy E and time t. But in the first instance we will consider E fixed and investigate only the diffusion in α. We define a pitch angle diffusion coefficient

$$D = \frac{(\Delta\alpha)^2}{t}$$

Then we can express

$$j = D\frac{\partial n}{\partial \alpha}$$

and

$$\text{div}\, j = \frac{\partial}{\partial \alpha}\left(D\frac{\partial n}{\partial \alpha}\right) = D\frac{\partial^2 n}{\partial \alpha^2}.$$

q is assumed to be zero after $t = 0$, so that finally we have $\partial n/\partial t = D\partial^2 n/\partial \alpha^2$. Setting $n(\alpha, t) = N(\alpha)\cdot T(t)$ and applying well-known procedures we obtain

$$n(\alpha, t) = \int_{-\infty}^{\infty} c(k)e^{ik\alpha - Dk^2 t}\, dk$$

as the most general solution. We define α as $\pi/2 - \theta$, and allow it to be negative and positive. We now use the initial condition: at $t = 0$, $n = n_0(\alpha)$ which can be expressed as

$$n_0(\alpha) = \int_{-\infty}^{\infty} c(k)e^{ik\alpha}\, dk$$

so that

$$c(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} n_0(\alpha')e^{-ik\alpha'}\, d\alpha'.$$

Substituting these expressions for $c(k)$ and integrating with respect to k we get

$$n(\alpha, t) = \frac{1}{(4\pi D)^{1/2}}\int_{-\infty}^{\infty} n_0(\alpha')e^{-(\alpha-\alpha')^2/4Dt}\, d\alpha'$$

We take $n_0(\alpha)$ to be constant over the limits $-\alpha_c < \alpha < \alpha_c$ and zero outside these limits. Then the solution can be written

$$n(\alpha, t) = \frac{C\alpha_c}{2}\left\{\phi\left[\frac{\alpha_c - \alpha}{(4Dt)^{1/2}}\right] + \phi\left[\frac{\alpha_c + \alpha}{(4Dt)^{1/2}}\right]\right\}$$
$$= C\alpha_c\left\{\phi\left[\frac{\alpha_c - \alpha}{(4Dt)^{1/2}}\right]\right\}$$

where $\phi(x)$ is the error function, defined as

$$\frac{2}{\sqrt{\pi}}\int_0^x e^{-\zeta^2}\, d\zeta$$

It is useful to examine the behavior of $n(0, t)$, i.e., the decrease with time in the concentration of particles having a pitch angle α_0 of 90° (and therefore confined to the equatorial plane).

$$n(0, t) = C\alpha_c\,\phi\left[\frac{\alpha_c}{(4Dt)^{1/2}}\right]$$

When $t = 0$, $[\alpha_c] \equiv x = \infty$, and $\phi[\alpha_c] = 1$. At first ϕ drops rapidly and after a time $t_{1/2}$ reaches the value 0.5. At that point $x \sim 0.5$ also, so that

$$x \sim 0.5 = \left[\frac{\alpha_c}{(4Dt_{1/2})^{1/2}}\right], \qquad \therefore\ t_{1/2} = \frac{\alpha_c^2}{D}$$

For $\phi \sim 0.1$, $x = 0.1$ also; $\therefore t_{0.1} = 100\alpha_c^2/4D = 25t_{1/2}$ These results for $t_{1/2}$ and $t_{0\cdot 1}$ are fairly representative. In greater detail we can consider the variation for other

values of α, essentially the function

$$\phi\left[\frac{\alpha_c - \alpha}{(4Dt)^{1/2}}\right].$$

A two-dimensional plot is given in Figure 6. It will be seen that the function ϕ decreases very rapidly for large values of α (close to α_c). This expresses the fact that after a long time particles will be confined closely to the equatorial plane (see Fig. 5).

If we define

$$N(t) = \int_0^{\alpha_c} n(\alpha, t) \cos \alpha \, d\alpha,$$

then $N(t)$ is given by the area under the various curves of Figure 7, weighted by a cosine function. The half time for $N(t)$ is seen to be somewhat less than $t_{1/2}$, but only by a factor at most.

To obtain a rough numerical estimate we use the value of D calculated earlier, i.e.,

$$D \cong k \frac{cd}{\gamma^2} \quad \text{(for relativistic electrons)}$$

where k is 10–20 depending on the nature of the atmosphere. Using $\gamma \sim 4$, and $d \sim 10^{-17}$ g/cm^3 we get $D \sim 3 \times 10^{-7}$ rad^2/sec. With $\alpha_c = 64°$, the particles must diffuse in pitch angle by about 26° or 0.5 rad. Therefore $t_{1/2} \sim 10^6$ sec or 2 weeks.

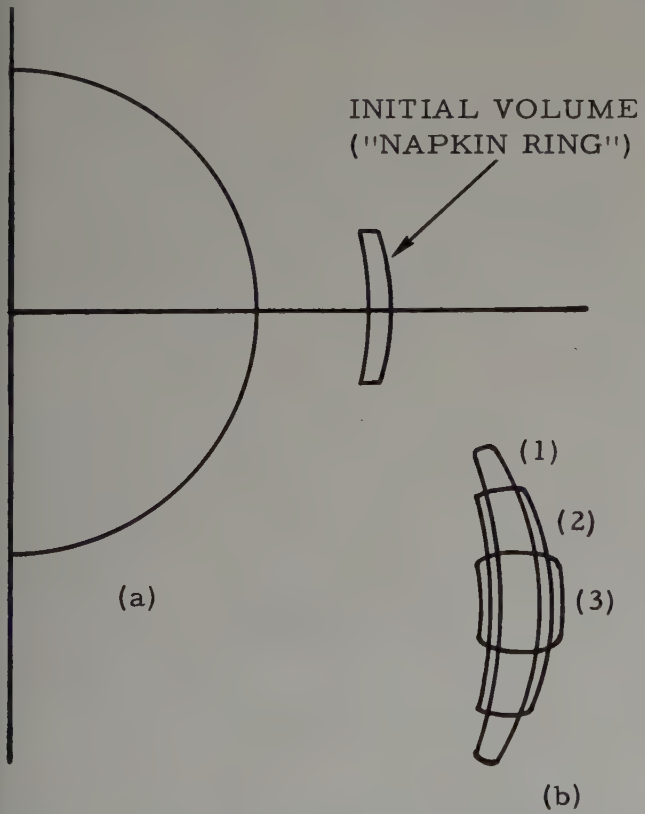

FIG. 6. The initial region into which the electrons diffuse is shaped like a napkin ring. It gradually approaches a circular cross section as electrons with small pitch angles are lost and only electrons with $\alpha \sim 90°$ remain. The exact dependence with time can be determined from the solution of a diffusion equation. See also Figure 7. Some radial diffusion of the electrons will also take place due to small angle scattering.

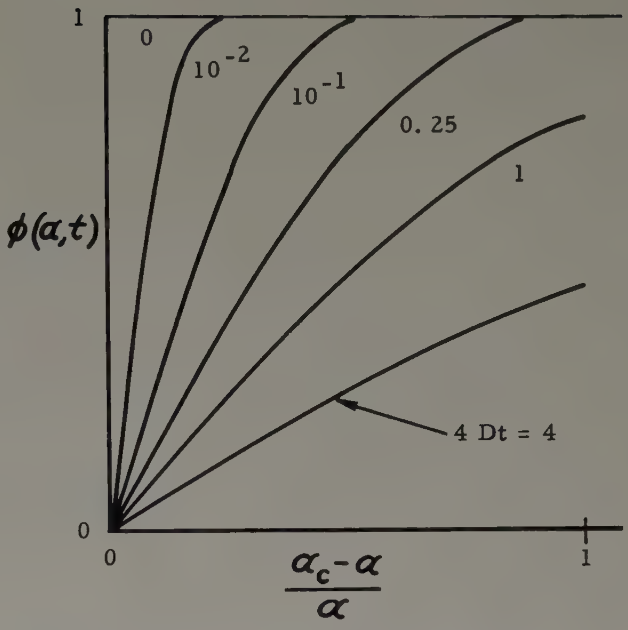

FIG. 7. Solution of the one-dimensional diffusion equation indicating the depletion of electrons as a function of their pitch angle.

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Encke's Method and Variation of Parameters as Applied to Re-entry Trajectories*,†

Robert M. L. Baker, Jr.‡

Abstract

The Encke and variation-of-parameters method of perturbation calculation are developed for the analysis of re-entry trajectories. The new techniques have the advantages of both higher accuracy and greater computational speed over the commonly employed re-entry calculations that involve integrations of all the forces acting on the vehicle (known to astronomers as Cowell's method). Applications to both re-entry from satellite and interplanetary orbits, which include lift as well as drag, will be mentioned.

Introduction

The principal deterrent to increased precision in trajectory analysis by numerical integration is the accumulation of end-figure error. If a problem formulation is employed that requires the integration of the total forces acting on a vehicle (a technique known to astronomers as Cowell's method) large end-figure errors may occur, and an additional penalty in calculation time is also paid. Even though the availability of large scale computing machinery tends to minimize this latter factor, the ability to control end-figure error becomes more in doubt as the number of numerical integration steps increases.

When perturbation methods are employed, one integrates only the departures from an analytic reference orbit, and the number of steps required for a given accuracy is drastically reduced, with a corresponding reduction in both end-figure error and computation time.

Two classes of perturbation methods are commonly employed for orbit determination by investigators familiar with celestial mechanics. The first of these, called "Encke's" method, utilizes a reference orbit, and deviations from this reference orbit resulting from perturbative forces not included in it are accumulated during the computational process. If the reference orbit selected is a conic section, for example, only the deviations of the force field from a central field need be integrated, and these lead to perturbative displacements from the reference conic.

An alternate approach involves the continuous correction ("rectification") of a reference orbit such that, at every instant, the reference orbit exactly conveys the position and velocity of the object. This technique is designated as the "variation of parameters" method. Suitable parameters for the 2-body Keplerian ellipse, which can be corrected continuously through the tra-

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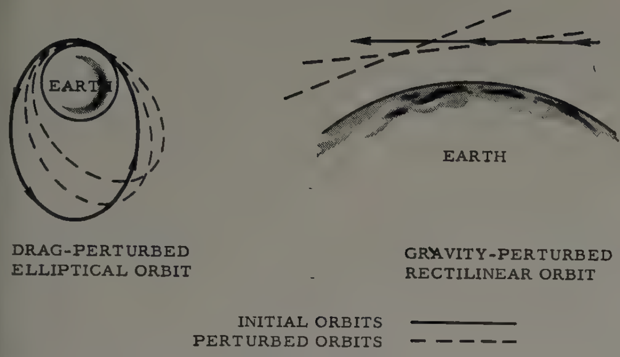

FIG. 1. Perturbation Techniques

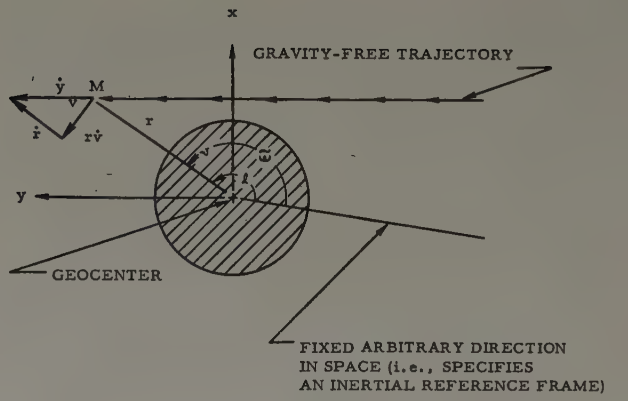

FIG. 2. Coordinate System

jectory, include eccentricity, semi-major-axis, and longitude of perigee.

The reference orbit that is to be perturbed is usually chosen to be the orbit that best depicts the gross motion of an object. All forces not taken into account in this gross motion are then categorized as perturbative forces. It is usually advisable to utilize as the reference orbit, an orbit that can be integrated analytically in closed form at the outset without requiring a step-by-step numerical procedure. These considerations immediately led astronomers to the utilization of the two-body Keplerian orbit as the reference orbit for the study of planetary perturbations. All forces exerted by drag, other planets, etc. are then incorporated as perturbative forces that tend to change the orbital parameters.

In the case of re-entry orbits, the drag forces predominate, and it is therefore advisable to discard the two-body orbit and substitute in its stead a purely gravity-free, drag orbit. Because all the drag forces are directed opposite to the velocity vector, the orbit will be rectilinear. Fig. 1 indicates the difference between the drag-perturbed and gravity-perturbed orbits.

Encke's Method

If we denote ξ and η as the x and y directed departures for the reference orbit, P_x and P_y as the x and y directed perturbative forces, and the zero subscript as the value of the quantity on the reference orbit, then Encke's method (in two dimensions) can be formulated (cf. (1)) as

$$\xi = \iint [(\ddot{x} - \ddot{x}_0) + P_x]\, dt^2 \qquad (1a)$$

$$\eta = \iint [(\ddot{y} - \ddot{y}_0) + P_y]\, dt^2 \qquad (1b)$$

where $\ddot{x}_0$ and $\ddot{y}_0$ represent the x and y components of acceleration that the re-entry body would have at time t, *if* it were moving on the reference orbit. Consider the coordinate system shown in Fig. 2. The rather unconventional labeling of the coordinate axes is employed in order to preserve the direction of the x-axis towards perigee.

Letting A_D represent the true drag acceleration, we find that

$$\ddot{x} = A_D \frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}}, \qquad \ddot{y} = A_D \frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \qquad (2a, b)$$

$$\ddot{x}_0 = 0, \qquad \ddot{y}_0 = (1/\dot{y}_0)(d\dot{y}_0/dy_0), \qquad (3a, b)$$

$$P_x = -x/r^3 + \dot{x}_A' + \dot{x}_L', \quad \text{and} \qquad (4a)$$

$$P_y = -y/r3 + \dot{y}A' + \dot{y}_L'. \qquad (4b)$$

where the A subscripts stand for perturbation that result from the asphericity of the Earth and the L subscripts stand for lift perturbations.

Since, in the reference orbit, all of the motion of the vehicle is directed along the y-axis, the basic drag equation of motion in this orbit can be given by

$$\ddot{y}_0 = -\left[\frac{C_{D_0} A_0 \rho_0 V_{c0}^2}{2 g_0 m_0}\right] \sigma\gamma\mu\alpha\dot{y}_0^{\,2} \equiv -D_0^{\,2}\sigma\gamma\mu\alpha\dot{y}_0^{\,2} \qquad (5)$$

where C_{D_0} is the reference value of the drag coefficient, A_0 is the initial projected frontal area of the vehicle, m_0 is the initial mass of the vehicle, g_0 is the acceleration of gravity at unit distance (979.81979 cm/sec^2), ρ_0 is the sea-level atmospheric density, $\sigma = \rho/\rho_0$, $\gamma = C_D/C_{D_0}$, $\mu = m_0/m$, $\alpha = A/A_0$, and $\dot{y}$ is the inertial speed of the vehicle with respect to the (fixed) atmosphere. Here ρ, C_D, A, and m respectively represent the true value of the atmospheric density, drag coefficient, area, and mass of the vehicle, while V_{c0} is the surface circular-satellite speed (7.905 km/sec). Actually, γ can be expected to be a function primarily of the three variables σ, $\dot{y}$ and T_s, the surface temperature of the vehicle. These variables will determine the transitional and continuum variation of the drag coefficient (2). In certain re-entry problems in which the transitional variation and compressibility variation of C_D (and, hence, of γ) occur at different points along the trajectory, it is possible to separate γ into two factors:

$$\gamma = \gamma(\sigma)\gamma(\dot{y}) \qquad (6)$$

The function $\gamma(\sigma)$ specifies the variation of the drag coefficient during the transition from free-molecule to continuum flow, and is strongly a function of the

local molecular mean-free path; consequently, of the density ratio σ. The function $\gamma(\dot{y})$ specifies the compressibility variation of C_D with the Mach number; hence, is properly a function of $\dot{y}$. If it is assumed that $\mu\alpha$ remains constant and equal to unity during re-entry* on the reference orbit and that the gross-variation of $\sigma\gamma(\sigma)$ can be obtained from a table or analytically by means of a power series, while $\gamma(\dot{y})$ can be approximately represented by

$$\gamma(\dot{y}) = A + B/\dot{y}, \tag{7}$$

then

$$\ddot{y}_0 = d\dot{y}_0/dt = -D_0^2[A + B/\dot{y}_0]\gamma(\sigma)\sigma\dot{y}_0^2 \tag{8}$$

The reference orbit can be computed in advance either numerically (or analytically through employment of Eq. (17)) and the value of Eq. (8) is predetermined for every point.

In the true orbit A_D must be computed by employing the exact *variations* of σ, γ, μ and α according to

$$A_D = -D_0^2\sigma\gamma\mu\alpha\nu^2, \tag{9}$$

where ν is the velocity of the re-entry body with respect to the atmosphere and, hence, must include the effects of cross-winds and the rotation of the Earth's atmosphere.†

Thus, equations 1a and 1b can be integrated and Encke's method carried out.

Variation of Parameters

In the case of the variation-of-parameters method discussed in (5), it is necessary to consider the rectilinear orbit only with respect to a spherically symmetrical Earth; therefore, the one angle, ϖ, and one distance, x, are the only parameters needed to specify the orbit. (Of course, considerations of the asphericity of the Earth and rotation of the atmosphere can be included also in the perturbation analysis.) Another quantity (e.g., y or v) is required to give the position of the object on this orbit. The orbital parameters, perigee distance, and the longitude of perigee, ϖ, vary only because of gravitational perturbative forces. Lift forces, normal to the trajectory, also can be easily included in the new perturbation scheme.

Let the grave symbol (`) denote the perturbative variation of a quantity with respect to time in which the variation results from gravity. Thus, the radial and tangential perturbative accelerations are:

$$\dot{r}^{`} = -1/r^2, \qquad r\dot{v}^{`} = 0 \tag{10}$$

where r is the radial distance and v is the true anomaly, i.e., the angle between r and the x-axis. From (5), we determine the equations that are to be integrated as follows:

$$\varpi = \int \varpi^{`}dt, \tag{11}$$

$$x = \int x^{`}dt, \tag{12}$$

$$\dot{y} = \int [\ddot{y} + \dot{y}^{`}]\, dt, \quad \text{and} \tag{13}$$

$$y = \int [\dot{y} + y^{`}]\, dt. \tag{14}$$

If $\sigma\gamma(\sigma)$ and $\gamma(\dot{y})$, respectively, can be represented by

$$\sigma\gamma(\sigma) = C_0 + C_2r^2 + C_4r^4 + \cdots \tag{15a}$$

$$= Y_0 + Y_2y^2 + Y_4y^4 + \cdots \tag{15b}$$

$$\text{and } \gamma(\dot{y}) = A + B/\dot{y} + \cdots \tag{16}$$

then we can establish the equation for $\int \ddot{y}\, dt$ as

$$\int \ddot{y}\, dt = [\dot{y}_i + B/A] \cdot \exp\left[-AD_0^2 \sum_{i=0}^{i=n} \frac{Y_{2i}y^{2i+1}}{2i+1}\Bigg|_{y_i}^{y}\right] - B/A. \tag{17}$$

Thus, given y (and the initial conditions $\dot{y}_i$ and y_i), $\int \ddot{y}dt$ can be explicitly determined in closed form.

As derived in (5),

$$x^{`} = xy/\dot{y}r^3, \tag{18}$$

$$\dot{x}^{`} = -x/r^3, \tag{19}$$

$$\dot{y}^{`} = -y/r^3 + \cdots \tag{20}$$

We add to the analytical integral, $\int \ddot{y}dt$, the numerically integrated value of $\int \dot{y}^{`}dt$. It should be noted that $\dot{y}^{`}$ can include also differences between the gross variation of, e.g., $\gamma(\dot{y})$ and its true variation, i.e., $\gamma(\dot{y}) - (A + B/\dot{y})$. In this sense, $\dot{y}^{`}$ will be equal to $-y/r^3$ plus other perturbative quantities. The sum of $\int \ddot{y}dt$ and $\int \dot{y}^{`}dt$ is then the true $\dot{y}$. In order to obtain y, we must add $y^{`}$ to $\dot{y}$ and then integrate, i.e.,

$$y = \int [\dot{y} - x^2/\dot{y}r^3]\, dt. \tag{21}$$

Following this procedure, the variation-of-parameters method can be carried out.

Applications and Conclusions

The perturbation and computational approaches of the foregoing sections are justified only if they lead to faster and more accurate calculations. Consequently, other calculations were performed in which the conventional integration of all the forces (Cowell's method)

* The variation of $\mu\alpha$ has been included, however, in (3).

† For example, in equatorial X, Y, Z coordinates, $\nu_X = \dot{X} + Y\dot{\theta}$, $\nu_Y = \dot{Y} - X\dot{\theta}$, $\nu_Z = \dot{Z}$, and $\theta = 2\pi$ (1.002,737,-803) Earth radii/mean solar day.

and the variation-of-parameters scheme were employed on the same problem. The problem chosen for analysis was the return to the Earth from an interplanetary journey and the Runge-Kutta integrational method coupled with the E. Adams correction procedure (4) was employed in the actual computations.

As discussed in (5), the conventional calculations (even though their equations were slightly simpler and more symmetric) required from 3 to 10 times more computer time than did the variation-of-parameters computations; furthermore, the many extra integrational steps involved in the non-perturbation methods caused a gradual accumulation of end-figure error.

Acknowledgements

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Optimum Exhaust Velocity Programming and Propulsion Efficiency*

Robert Fox†

Abstract

The variational problem of most efficient utilization of the fuel is considered for the case of a rocket with separate fuel and energy sources and which operates in field-free space. It is found that the exhaust velocity should be low (but not zero) initially and should rise, during powered flight, in a manner described in the text. This solution leads to a natural definition of propulsion efficiency for such a system which is invariant to choice of inertial reference frame. Some properties of the variational solution are discussed briefly.

Introduction

"Propulsion efficiency" is a term which does not appear at present to have a well-defined meaning or to be well understood when applied to the topic of rocket propulsion. For present-day chemical rockets, the term has little significance since maximum performance usually results by maximizing the energy content per unit mass of the propellant if the density is reasonably high and its chemical properties render it compatible with the operational requirements of the system. This situation results from the fact that the energy available resides within the propellant and has an upper limit set by the physics of molecules. Also, there is an upper limit to the mass ratio of such rockets since the propellant is carried in a container.

Propulsion systems applicable to travel between the planets may not have these limitations. An example is an ion propulsion system utilizing nuclear power and cesium propellant. Here the energy source is independent of the propellant. On the other hand, even though nuclear energy is being used the energy source is not infinite and one must consider the problem of using the device in the most efficient manner possible. Thus we are led to the consideration of "propulsion efficiency".

We shall find that the problem is a dual one. There is the fundamental problem of distributing the available energy over the propellant mass (variation of exhaust velocity) in such a way as to maximize the total velocity change. We shall assume that the exhaust velocity can be varied without affecting the propulsion system mass. In addition, if we are to talk about "propulsion efficiency" we must find a unique and meaningful definition for this quantity.

Optimization of the Exhaust Velocity

Let us consider first the problem of variation of the exhaust velocity in the optimum manner for the ideal

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case of a rocket operating in a space with no gravitational field. As we shall see, the "obvious" solution, letting the velocity v of the exhaust gases with respect to the rocket be equal to the instantaneous rocket velocity V, is not, in general, the correct one.

Let the initial mass M_1 of the rocket under consideration be composed of a fuel propellant mass M_f and a dead mass M_s which includes the propulsion machinery, power plant, control equipment, and the like. Let the initial velocity be V_1 in some inertial frame of reference. Let E be the total available energy which can be given to the exhaust gases. This is less than the thermal energy produced by the power plant since most of the thermal energy ends up as rejected heat energy in the power plant and propulsion system. We want to program the exhaust velocity v (measured relative to the rocket) according to the mass of propellant m which has already been consumed. Then, the momentum equation is

$$(M_1 - m)\,dV = v(m)\,dm \qquad M_1 = M_s + M_f. \quad (1)$$

The final rocket velocity V_2 is

$$V_2 = V_1 + \int dV = V_1 + \int_0^{M_f} \frac{v(m)\,dm}{M_1 - m}. \quad (2)$$

$v(m)$ is still an arbitrary function and V_2 is a function of this function. We want to find the maximum value of V_2 subject to the auxiliary condition

$$2E = \int_0^{M_f} v^2(m)\,dm = \text{constant}. \quad (3)$$

This expresses the limitation on available energy. We have implied that the efficiency of the power plant and propulsion equipment for converting thermal energy to exhaust gas kinetic energy is independent of v. The solution of the variational problem posed by equations (1), (2), is obtained by solving the Euler equation[1]

$$\frac{\partial}{\partial v}\frac{v(m)}{M_1 - m} + \lambda \frac{\partial}{\partial v} v^2(m) = 0; \quad (4)$$

λ is an undetermined parameter which may be evaluated by using (3). From (4) we have

$$v(m) = -\frac{1}{2\lambda(M_1 - m)}. \quad (5)$$

Substituting (5) into (3) we find for λ,

$$\lambda = -\sqrt{\frac{M_f}{8EM_1(M_1 - M_f)}},$$

so that

$$v(m) = \frac{1}{M_1 - m}\sqrt{\frac{2EM_1(M_1 - M_f)}{M_f}} = \frac{1}{M_1 - m}\sqrt{\frac{2EM_2\mu}{\mu - 1}} \quad (6)$$

This is the velocity program which maximizes the final velocity under the conditions outlined. If we denote by v_0 the constant exhaust velocity obtained by distributing the energy E uniformly over the propellant mass

$$\bar{v} = \sqrt{\frac{2E}{M_f}},$$

then (6) may be expressed

$$v(m) = \frac{\bar{v}}{\sqrt{\mu}}\left(\frac{M_1}{M_1 - m}\right), \quad (7)$$

where the mass ratio μ is

$$\mu = \frac{M_1}{M_2}. \quad (8)$$

Note that the initial and final values of $v(m)$ are

$$v(0) = \frac{\bar{v}}{\sqrt{\mu}}, \qquad v(M_f) = \sqrt{\mu}\,\bar{v}. \quad (9)$$

Also note from the very nature of equations (2), and (3) that the result is invariant to a transformation from one inertial frame to another since only V_2, V_1 are affected by such a transformation and the change in rocket velocity $V_2 - V_1$ is invariant. V does not appear inside the integrals. Substituting (6) into (2) we have for the velocity change

$$V_2 - V_1 = \sqrt{\frac{2EM_f}{M_1(M_1 - M_f)}} = \sqrt{\frac{2E(\mu - 1)}{M_2\,\mu}} \quad (10)$$

since $M_2 = M_1 - M_f$. This may be rewritten

$$\frac{M_2}{2}(V_2 - V_1)^2 = \frac{\mu - 1}{\mu}E \quad (11)$$

if the exhaust velocity v is programmed according to (6). Any other exhaust velocity program would result either in reduced velocity change or in different E, μ (propellant mass) requirements. For example, if we choose instead $v(m) = \bar{v} = $ constant, then

$$V_2 - V_1 = \bar{v}\ln\mu,$$

and the relation between M_2, $(V_2 - V_1)$, μ, E is

$$\frac{M_2}{2}(V_2 - V_1)^2 = \frac{\ln^2\mu}{\mu - 1}E. \quad (12)$$

In this case, $(V_2 - V_1)$ is less than that given by (11) except in the trivial case $\mu = 1$.

Now let us investigate the connection between the velocity program (6) and the one obtained by letting the exhaust velocity equal the rocket velocity. In this case the exhaust gas velocity as measured in the same inertial frame as the rocket velocity is zero, since we are assuming that the exhaust gases are directed toward the rear. Intuition tells us that this ought to be the best solution, insofar as utilization of the energy is concerned. The rocket equation now becomes

$$\frac{dV}{v} \equiv \frac{dV}{V} = \frac{dm}{M_1 - m}. \quad (13)$$

The solution is

$$\frac{V_2}{V_1} = \mu. \tag{14}$$

Note the singularity for $V_1 = 0$.

Let us also calculate the energy consumed. We have:

$$E = \tfrac{1}{2} \int_0^{M_f} v^2(m)\, dm.$$

Now,

$$v(m) = V(m) \quad \text{and} \quad \frac{V(m)}{V_1} = \frac{M_1}{M_1 - m}, \quad \text{from (14)}.$$

Therefore,

$$\begin{aligned} E = \frac{V_1^2 M_1^2}{2} \int_0^{M_f} \frac{dm}{(M_1 - m)^2} &= \frac{V_1^2 M_1}{2} (\mu - 1) \\ &= \frac{M_2 V_2^2}{2} \frac{\mu - 1}{\mu} \end{aligned} \tag{15}$$

or, since $V_2 = \mu V_1$,

$$E = \frac{(V_2 - V_1)^2}{2} \frac{\mu}{\mu - 1} M_2. \tag{16}$$

We see that this equation is identical with (10) from the variational solution. There is, however, a distinction. Here we have the auxiliary condition $V_2/V_1 = \mu$ from (14), so that for a given value of the initial velocity V_1 and the mass ratio μ, both E and V_2 are determined. In the variational case E was arbitrary. Thus, the velocity program $v = V$, with $\mu = V_2/V_1$ as an auxiliary condition, is a particular solution of the variation problem but is not the general solution for arbitrary E.

A further difficulty with the program $v = V$ is that it depends upon the choice of inertial frame in which V is measured. Observers in two different inertial frames a and b could not simultaneously apply such a program to a given rocket. Using these two observers, we can now elucidate further the connection between the variational solution and the program $v = V$.

Consider the variational problem for the case $V_1^a = 0$ as measured by an observer in an inertial frame a. Let the mass ratio be μ and the available energy E. Now transform the observer (not the rocket) to an inertial frame b so that the observer then sees

$$\frac{V_2^b}{V_1^b} = \mu.$$

The relative velocity of the two inertial frames is

$$\bar{V} = V_1^b - V_1^a = V_1^b = \frac{V_2^a}{\mu - 1}, \tag{17}$$

since $V_2^a - V_1^a \equiv V_2^b - V_1^b$.

In frame a the initial value of the exhaust velocity is, from (6),

$$v^a(0) = \frac{1}{M_1} \sqrt{\frac{2EM_1(M_1 - M_f)}{M_f}}.$$

Substituting for E from (10),

$$v^a(0) = \frac{M_1 - M_f}{M_f} V_2^a = \frac{V_2^a}{\mu - 1}.$$

The value of the exhaust velocity as measured by the observer in frame b is then

$$v^b(0) = v^a(0) - \bar{V} = 0.$$

Furthermore, by substituting (6) into (1) it can readily be shown that the value of the exhaust velocity seen by an observer in frame a (not an observer on the rocket) is constant during the entire burning period. Thus, the variational solution for optimum use of the energy and starting from rest in frame a is physically identical to the solution $v = V$ in the *other* inertial frame b which is related to a through (17).

Propulsion Efficiency

We are now in a position to discuss the concept of propulsion efficiency as applied to rockets. To begin, we might ask what properties such a quantity should have. It has already been shown how one can obtain the maximum velocity change in a rocket for a given mass ratio and available energy. Clearly, since the velocity change one can obtain in a rocket is the prime index of its performance (for missions in field-free space), this solution should correspond to maximum propulsion efficiency. Further, since the solution we obtained is invariant to a transformation from one inertial frame to another, the propulsion efficiency must also have this property. Third, the propulsion efficiency should never be negative or exceed unity. To summarize, using the symbol η to designate propulsion efficiency:

1. Maximum η corresponds to the variational solution (6).
2. η is invariant to translation of the observer from one inertial frame to another.
3. $0 \leqq \eta \leqq 1$.

From equation (10) we can immediately find a quantity which satisfies all these requirements, namely,

$$\eta = \frac{M_2(V_2 - V_1)^2}{2E}. \tag{18}$$

For the optimum exhaust velocity program, we have from (10)

$$\eta = \frac{\mu - 1}{\mu}. \tag{19}$$

Clearly, with this definition $\eta \leqq 1$, since this exhaust velocity program maximized $(V_2 - V_1)$, and since M_2 and E are fixed quantities. Further, $\eta \geqq 0$ for any exhaust velocity program since all the quantities involved are positive definite. The invariance is obvious. Also, the problem of finding an exhaust velocity program that maximizes η is identical with the variational problem stated earlier.

It is interesting to consider three other possible defi-

nitions for η, which shall be designated η_1, η_2, and η_3, and investigate the conditions under which they fail.

$$\eta_1 = \frac{E_1^k - E_2^k}{E} = \frac{1}{2}\left(\frac{M_1 V_1^2 - M_2 V_2^2}{E}\right) \quad (20)$$

$$\eta_2 = \frac{M_2(V_2^2 - V_1^2)}{2E} \quad (21)$$

$$\eta_3 = \frac{\text{rate of doing work}}{\text{rate of total energy loss}}. \quad (22)$$

Intuitively, (20) would appear to be more reasonable than (18). In fact, if $V_1 = 0$, conditions 1, 3 above are satisfied. However, condition 2 is not. Further, if $V_1 = 0$, condition 3 is not always satisfied. For example, consider a rocket with constant exhaust velocity $\bar{v}$. Then,

$$V_2 - V_1 = \bar{v} \ln \mu. \quad (23)$$

Let

$$\mu = \frac{M_1}{M_2} < \left(\frac{V_2}{V_1}\right)^2. \quad (24)$$

Substituting (24) into (20) we have

$$\eta_1 < 0$$

for such cases.

It may be observed that if $V_1 = 0$ in an inertial frame b and if E is related to M_1, V_1, μ by (15) so that $v = V$ is the solution of the variational problem in this frame, then we find $\eta_1 = 1$. This is to be expected since the exhaust gases have zero kinetic energy in this frame of reference.

The definition η_2 does not have the deficiency of becoming negative. However, for the particular case just discussed we find that $\eta_2 > 1$ since the negative term is reduced by the factor $1/\mu$ so that

$$\eta_2 > \eta_1 = 1.$$

Finally, let us consider the definition which may be found in certain textbooks[2], namely, η_3.

We have, for an element dm of propellant expended,

$$\eta_3 = \frac{v \dfrac{dm}{dt} V}{\dfrac{1}{2}\dfrac{dm}{dt} v^2 + \dfrac{1}{2}\dfrac{dm}{dt} V^2} = \frac{Vv\,dm}{\frac{1}{2}(dmv^2 + dmV^2)} \quad (25)$$

The numerator is just force $\times$ velocity $\times$ time element. The two terms in the denominator are the energy given to the propellant and the diminution of rocket kinetic energy due to the reduction in propellant mass. Note that the three terms all have different transformation properties. Clearly, $\eta_3 < 0$ in any reference frame in which $V < 0$.

Conclusions

It has been shown that for the case of no gravitational field the performance of a rocket can be improved by varying the rocket exhaust velocity in the appropriate manner during the burning period if the energy source and propellant source are distinct from one another. Note that the *rate* of burning, or the thrust, does not enter into the discussion. The problem of optimum thrust programming for real missions in a gravitional field is a quite different and considerably more difficult problem than the idealized one considered here. The foregoing discussion does indicate, however, that for missions involving propulsion devices such as ion rockets it may be worth while to consider varying the exhaust velocity as well as the thrust. For example, if one is going "uphill" against the sun's gravitational field and the propulsion system is strongly power-limited, the velocity program (6) helps in two ways. First, since the initial exhaust velocity is lower, the initial thrust can be higher, resulting in greater initial acceleration and reduced flight time. Second, because of the lower initial exhaust velocity and the power limitation, the initial propellant consumption rate is also higher so that on the average the propellant is dumped at a lower gravitational potential.

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2. *Other Main Headings*. The paper can be divided into principal sections as appropriate. Headings or paragraphs are not numbered. Secondary headings are in caps and lower case; they are flush left and underlined.
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$$(a^2m_0/at_2^2)(a^2m_0/as_2^2) = 0 \qquad (1)$$

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   1. C. S. Draper, Education in the Astronautical Sciences, *J. Astronautics,* Vol. IV, No. 2, pp. 29–30, 1957